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Stability and Application of an Orbit-Averaged Magneto-Inductive Particle Code*

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A new linear stability analysis of an orbit-averaged magneto-inductive particle code is given. The analysis demonstrates the stability of this explicit algorithm at large time steps that would violate the usual Courant condition on Alfvén wave propagation in a conventional algorithm. Application of the particle code to the simulation of a mirror-machine experiment using realistic parameters is described.

1. INTRODUCTION

In [1] an orbit-averaged, magneto-inductive particle simulation algorithm was introduced. This algorithm corresponds to a physical model in which electrons are tied to magnetic field lines and are presumed to charge-neutralize large-orbit ions whose cross-field motion leads to diamagnetic currents, significant self-magnetic fields, and inductive electric fields [2, 3]. Analysis of the numerical properties of the magneto-inductive scheme was made in [1] only in the limit of no orbit averaging. Here we shall present a numerical stability analysis of a simple orbit-averaged, magneto-inductive algorithm (Section 2) and give some further simulation examples (Section 3). We shall prove that an *explicit* predictor–corrector scheme is stable at large time step. This is in marked contrast to the electrostatic models in the companion paper, which require implicit-time differencing to achieve stability at large time step [4].

2. STABILITY ANALYSIS OF MAGNETO-INDUCTIVE MODEL

We simplify the stability analysis by considering the following one-dimensional slab model. The particles are advanced by use of

$$v_x^{n+1/2} = v_x^{n-1/2} + qB_z^* \Delta t(v_y^{n+1/2} + v_y^{n-1/2})/2mc, \tag{1}$$

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$$v_y^{n+1/2} = v_y^{n-1/2} + qE_y^* \Delta t/m - qB_z^* \Delta t(v_x^{n+1/2} + v_x^{n-1/2})/2mc, \tag{2}$$

$$x^{n+1} = x^n + v_x^{n+1/2} \Delta t. \tag{3}$$

The fields are determined by

$$(D_x^2/2)[(1+\varepsilon) A_y^{M+1} + (1-\varepsilon) A_y^M] = -(4\pi/c)\langle J_y \rangle^{M+1/2}, \tag{4}$$

$$E_y^{M+1} = -(c\Delta T)^{-1}(A_y^{M+1} - A_y^M), \tag{5}$$

$$B_z^{M+1} = D_x A_y^{M+1}, \tag{6}$$

where ε is a centering parameter, $0 \leqslant \varepsilon \leqslant 1$, $\langle J_y \rangle^{M+1/2}$ is the orbit-averaged current density,

$$\langle J_y \rangle^{M+1/2} = \frac{1}{N+1} \sum_{j=0}^{N} \sum_i \frac{q}{2} [S(x_i^j - x) + S(x_i^{j+1} - x)]\, v_{y,i}^{j+1/2}, \tag{7}$$

with $\Delta T = N\Delta t$ and $0 \leqslant j \leqslant N$ corresponds to the interval $M\Delta T \leqslant t \leqslant (M+1)\Delta T$, and

$$\begin{aligned} (E_y, B_z)^* &= (E_y, B_z)^M \qquad \text{(predictor)} \\ &= (E_y, B_z)^{M+1} \qquad \text{(corrector)} \end{aligned} \tag{8}$$

with interpolation from the spatial grid to the particle position using $S(x_i^j - x)$.

We assume that the plasma is cold, uniform, and immersed in an externally applied, uniform magnetic field B_0. We have analyzed the propagation characteristics of small-amplitude oscillations supported by the difference equations (1)–(8). The velocities at a time $l\,\Delta t$ after the beginning of a macro-time interval are related to the initial velocities by

$$\begin{aligned} \mathbf{v}^{l+1/2} \equiv (v_x, v_y)^{l+1/2} &= \frac{\mathbf{\Theta}^{2l}}{|\Theta|^l} \cdot \mathbf{v}^{1/2} + \sum_{j=1}^{l} \frac{\mathbf{\Theta}^{2j+1}}{|\Theta|^l} \cdot \frac{q\Delta t\, \mathbf{E}^*}{m} \\ &= \left(\mathbf{I} - \frac{\mathbf{\Theta}^2}{|\Theta|}\right)^{-1} \cdot \left(\mathbf{I} - \frac{\mathbf{\Theta}^{2l}}{|\Theta|^l}\right) \cdot \frac{\mathbf{\Theta}}{|\Theta|} \cdot \frac{q\,\Delta t\, \mathbf{E}^*}{m} + \frac{\mathbf{\Theta}^{2l}}{|\Theta|^l} \cdot \mathbf{v}^{1/2}, \end{aligned} \tag{9}$$

with

$$\mathbf{\Theta} = \begin{pmatrix} 1 & \omega_c \Delta t/2 \\ -\omega_c \Delta t/2 & 1 \end{pmatrix}, \tag{10}$$

where $\omega_c = qB_0/mc$, and $|\Theta| = \det \mathbf{\Theta} = 1 + \omega_c^2 \Delta t^2/4$. Equation (9) was obtained by summing a geometric series. The orbit-averaged, linearized current is calculated similarly

$$\langle J_y\rangle^{M+1/2} = \frac{1}{N+1}\sum_{j=0}^{N} q n_0 v_y^{j+1/2}$$
$$= \hat{e}_y \cdot \left(\mathbf{I} - \frac{\mathbf{\Theta}^2}{|\Theta|}\right)^{-1} \cdot \left(\mathbf{I} - \frac{\mathbf{\Theta}^{2N+2}}{|\Theta|^{N+1}}\right) \frac{q n_0 \mathbf{v}^M}{N+1}$$
$$+ \hat{e}_y \cdot \left(\mathbf{I} - \frac{\mathbf{\Theta}^2}{|\Theta|}\right)^{-1} \cdot \left[\mathbf{I} - \frac{1}{N+1}\left(I - \frac{\mathbf{\Theta}^2}{|\Theta|}\right)^{-1} \cdot \left(\mathbf{I} - \frac{\mathbf{\Theta}^{2(N+1)}}{|\Theta|^{(N+1)}}\right)\right]$$
$$\cdot \frac{\mathbf{\Theta}}{|\Theta|} \cdot \frac{q^2 \Delta t\, n_0 E_y \hat{e}_y}{m}, \tag{11}$$

where $\mathbf{v}^M$ is the velocity vector at $t = M\Delta T + \Delta t/2$.

We can use Eq. (9) to relate $\mathbf{v}^{M+1}$ to $\mathbf{v}^M$ and E_y^* by taking $l = N \equiv \Delta T/\Delta t$. The system of equations is then closed with Eqs. (4), (5), (8), and (11), which define the perturbed current and E_y^*, and relate the perturbed fields to the current. As is obvious from Eqs. (9)–(11), there is a considerable amount of algebra involved in this analysis, but the methodology is the same as for the derivations encountered in companion paper [4]. The primary cause of the increased complexity in this calculation is the $\mathbf{v} \times \mathbf{B}$ rotation in the particle velocity advance.

The dispersion relation is a quartic equation in $z \equiv \exp(-i\omega\Delta T)$. In the limit $\omega_c^2 \Delta T^2 \ll 1$, we have analytically recovered two solutions corresponding to compressional Alfvén waves

$$\omega^2 = k^2 v_A^2/(1 + k^2c^2/\omega_p^2), \tag{12}$$

where v_A is the Alfvén speed $v_A = c\omega_c/\omega_p$. The remaining normal modes are nonphysical. The Alfvén waves are neutrally stable for $\omega_p^2/k^2c^2 \ll 1$ independent of ε in Eq. (4). For finite ω_p^2/k^2c^2, the Alfvén waves acquire some damping. One of the unphysical modes is a Nyquist oscillation that is neutrally stable for $\varepsilon = 0$ and all

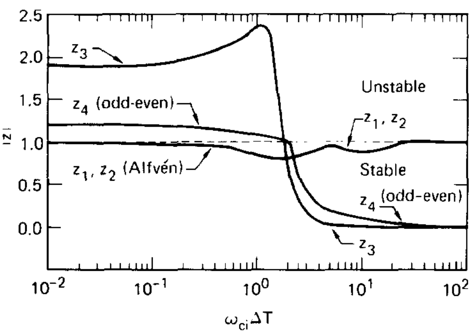

FIG. 1. Amplification factor $|z|$ vs $\omega_{ci}\Delta T$ for cold plasma normal modes in the explicit, orbit-averaged magneto-inductive algorithm for $(\omega_p/kc)^2 = 3$, $\varepsilon = 1$ in Eq. (4), and $\Delta T/\Delta t = 10^3$. There are four roots to the quartic dispersion relation, all of which are stable for $\omega_{ci}\Delta T \gg 1$.

values of ω_p^2/k^2c^2. The remaining unphysical normal mode is purely growing for $\omega_c^2 \Delta T^2 \ll 1$ and $\omega_p^2/k^2c^2 > 1$ and heavily damped for $\omega_p^2/k^2c^2 \ll 1$. We have used these analytical solutions to check numerical solution of the dispersion relation, which is then employed for larger time steps (see Fig. 1 for solutions with $\varepsilon = 1$ and $\omega_p^2/k^2c^2 = 3$).

For $\omega_c^2 \Delta T^2 \gg 1$ and $\varepsilon > 0$, all the modes are *stable*. For $\varepsilon = 0$, $\omega_p^2/k^2c^2 > 1$, and $\omega_c^2 \Delta T^2 \gg 1$, the Alfvén waves are weakly unstable. The two unphysical modes are both damped Nyquist oscillations for $\omega_c^2 \Delta T^2 \gg 1$, $\varepsilon > 0$, and all values of ω_p^2/k^2c^2. The schematic of the dispersion relation for Alfvén waves exhibited in Fig. 2 of [1] is qualitatively correct. In the absence of orbit-averaging and other numerical effects, the frequency of the Alfvén branch would have asymptotically approached the

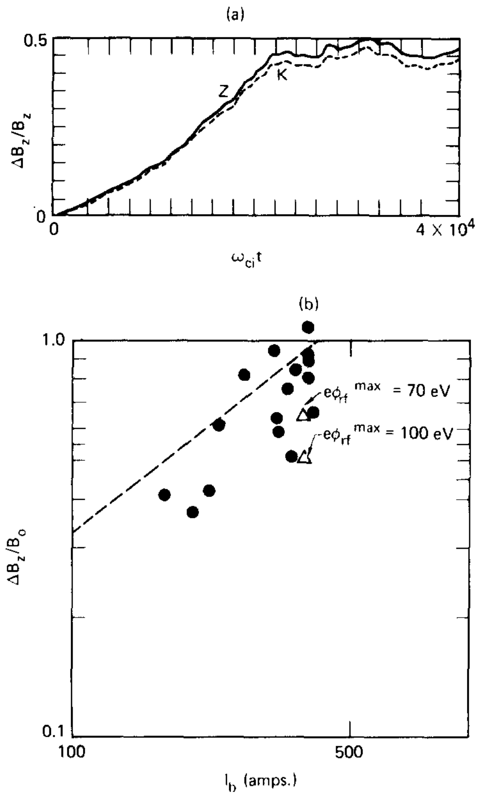

FIG. 2. (a) SUPERAVERAGE simulations of 2XIIB buildup to high β, showing field reversal dianostics $Z \equiv 1 - (B_z)_{min}/(B_z)_{max}$ and $K \equiv 1 - (B_z)_{min}/B_0$ in the midplane ($z = 0$), maximized with respect to radius. (b) Comparison of $\Delta B_z/B_0$ from SUPERLAYER and SUPERAVERAGE simulations of 2XIIB buildup to $\Delta B/B_0 \lesssim \mathcal{O}(1)$ inferred from 2XIIB experimental data (3/14/77) for vacuum magnetic field $B_0 = 4350$ G from [6]. The symbol $e\phi_{rf}^{max}$ indicates the peak wave amplitude used in the quasi-linear model of ion cyclotron turbulence. - - -, Superlayer; Δ, Superaverage; ●, 2XIIB data (4/14/1977).

cyclotron frequency for large k in accord with Eq. (12). However, as a consequence of orbit averaging and the use of $\varepsilon > 0$ in Eq. (4), the Alfvén frequency at large k and for $\omega_c^2 \Delta T^2 \gg 1$ approaches $\mathcal{O}(\pi/\Delta T)$ and acquires some damping. Thus, that part of the Alfvén branch in the frequency interval $(\pi/\Delta T)^2 < \omega^2 < \omega_c^2$ that would violate the usual stability constraint in a conventional explicit code [wherein $kv_A \Delta T < \mathcal{O}(1)$ for stability] has been suppressed, and no instability is encountered even though $kv_A \Delta T \gg 1$.

It is an unexpected and perhaps a fortuitous characteristic of the magneto-inductive model that stability at large ΔT is possible with an explicit predictor–corrector scheme, in contrast to an electrostatic model that always requires implicit differencing. For the same magneto-inductive scheme, there is instability for $\omega_c^2 \Delta T^2 \leqslant 1$ and $\omega_p^2/k^2c^2 > 1$. The stability characteristics of the explicit magneto-inductive algorithm have been confirmed in simulations reported in [1] and in the next section.

3. Simulations

Orbit averaging in a two-dimensional magneto-inductive algorithm has been very successful [1]. This is due to the stability of the algorithm for both the solution of the field equations with $\omega_c \Delta T \gg 1$ and the advancement of the particles with $\omega_c \Delta t < 1$. This separation of time scales allows us to orbit average over the particle trajectories (in the cases studied so far, this means averaging over many ion cyclotron and axial bounce periods in a magnetic mirror), and to greatly reduce the number of required particles. Simulations with one- and two-dimensional orbit-averaged, magneto-inductive codes are consistent with our stability analysis and confirm the calculated sensitivities to the centering parameter ε in Eq. (4), to $\omega_c \Delta T$ and to ω_p/kc.

An impressive example of orbit averaging is given by our simulations of the formation of a high β ($\beta \equiv 8\pi n T_i/B_0^2$) neutral-beam-injected mirror plasma. Two-dimensional simulations of beam-injected, axisymmetric mirror plasmas without orbit averaging have been described previously by Byers [3] and with orbit averaging in [1]. In these models electrons provide a warm, charge-nautralizing background. Electrostatic fields are forced by open field-line physics to be relatively weak for electron temperatures ($T_e \sim 100$ eV) that are much less than the mean ion energies ($\varepsilon_i \sim 13$ keV) and $-\nabla\phi$ is modeled as a fixed ambipolar electric field conforming to those typically observed in mirror plasmas, $\Delta\phi \approx T_e \ln(n_i/n_0)$. The simulation realistically models charge exchange and ionization of a neutral beam, electron-ion drag, and quasi-linear velocity diffusion due to drift-cyclotron loss-cone instability [5]. Warm streaming ions injected axially and hot ions coming from neutral beams are described with Newton–Lorentz equations of motion, and self-consistent, magneto-inductive fields are determined by solution in cyclindrical coordinates (r, z) of Ampere's and Faraday's equations, ignoring transverse and longitudinal displacement currents. Reference [1] and Eqs. (1)–(8) present the basic form of the orbit-averaged difference equations.

For these simulations we have been able to use parameters directly corresponding to the Lawrence Livermore National Laboratory 2XIIB experiment that attempted to achieve field reversal [6]. These simulations were able to simultaneously resolve the ion cycotron time scale in the vacuum magnetic field $\omega_{ci} = 2.8 \times 10^7\ s^{-1}$, and the ion-electron slowing down rate $\nu_s^{i/e} = 3 \times 10^2\ s^{-1}$ without artificial distortion. Steady states were achieved at times corresponding to several milliseconds into the experiment with 300 to 400 A of neutral-beam injection sustaining the plasma against losses due to drag, quasi-linear diffusion, and loss of adiabaticity [7].

The emergence of significant particle losses due to non-conservation of magnetic moment was the most important new simulation result. Previous simulations of 2XIIB [3] did not exhibit these losses, because the drag and injection rates were artificially accelerated ($\times 10^2$) to complete the simulation in a reasonable amount of computer time; in consequence, because of the foreshortened particle lifetimes, ions completed many fewer axial bounces than achieved in the experiment and than are required for much loss of adiabaticity to occur. The new simulations

(a) required 500 to 1000 ions rather than the 20,000 previously needed,

(b) had a correspondingly smaller memory requirement, and

(c) were able to run 10 to 100 more time steps in the same amount of computer time.

The principal physical consequences of the increased axial losses in the new simulations are both a downward revision of the predictions of field reversal $\Delta B_z/B_z$ obtained in earlier SUPERLAYER simulations [6] to values significantly less than unity (see Fig. 2) and a lengthening of the hot plasma in the simulation, which has always been shorter than was experimentally observed.

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